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The effect of a containing cylindrical boundary on the velocity of a large gas bubble in a liquid

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The velocity of a large gas bubble rising along the axis of a cylindrical container filled with liquid, is derived to a first approximation from a simple flow model of the system. The resulting expressions involve ratios of certain infinite series, which are evaluated for a few cases, and the theory is then seen to agree with previously accepted results at its asymptotes. Experiments are performed which confirm the first approximation and which allow the theory to be recast as a semi-empirical theory relating bubble volume with velocity. In this form it agrees with the results of Uno & Kintner (1956), but indicates that the relation between volume and curvature which they employed was in error.

1. Introduction

An introduction to previous work on large gas bubbles in liquids begins inevitably by referring to the papers by Davies & Taylor (1950) and Dumitrescu (1943). Davies & Taylor studied an infinite liquid in which the bubble assumes the characteristic spherical-cap form shown in figure 1 (plate 1). The rim of this cap subtends an angle of approximately 100° at the apparent centre of curvature, while the base of the bubble is unsteady and fluctuates about a plane through the rim. Their resulting theoretical expression for bubble velocity was

$$U_\infty = \tfrac{2}{3}(g\bar{a})^{\frac{1}{2}}, \tag{1}$$

where g is the gravitational acceleration and $\bar{a}$ the apparent (or mean) radius of curvature of the bubble cap. At the other extreme, Dumitrescu's analysis applied to the infinitely long bubble in a cylindrical tube of circular cross-section, the type of bubble eventually produced when a vertical liquid-filled column is suddenly opened at its lower end. For this problem his analysis gave

$$U_s = 0{\cdot}496(gb)^{\frac{1}{2}}, \tag{2}$$

where b is the radius of the tube. There is ample evidence to support the forms exhibited by equations (1) and (2), although experimental values for the numerical constants differ slightly. For example, the mean value of $U_\infty/(g\bar{a})^{\frac{1}{2}}$ from the experiments of Davies & Taylor was 0·655, while Rosenberg's (1950) experiments on the same problem indicated a value of 0·645. In examining this slight difference, Collins (1966, subsequently termed I) has shown that one must regard equation (1) as a good first approximation, so good in fact that it is within 2 %

of a second approximation which was determined using a perturbation technique to be

$$U_\infty = 0{\cdot}652(g\bar{a})^{\frac{1}{2}}, \tag{3}$$

and which is in excellent agreement with the experimental evidence. As a result, the first approximation is expected to give a good description of bubble velocity in more complicated geometries where a second approximation would be very involved. In essence the first approximation satisfies the requirement that the gas pressure within the bubble should be constant, only to the first order in the neighbourhood of the front stagnation point. The present analysis constitutes a first approximation.

Dumitrescu's experiments showed that when the cylinders were sufficiently large so that surface tension and viscous forces were insignificant, as all theories considered here require, then $U_s/(gb)^{\frac{1}{2}} = 0{\cdot}49$. Experiments for the same problem were also performed by Davies & Taylor who found that, for their largest tube, $U_s/(gb)^{\frac{1}{2}}$ ranged from 0·466 to 0·49. Their objective was to examine their own theoretical result for this problem which was

$$U_s = 0{\cdot}464(gb)^{\frac{1}{2}}, \tag{4}$$

and which they themselves regarded as a very rough estimate. Nicklin, Wilkes & Davidson (1962) have confirmed Dumitrescu's experimental results and have found that (2) also describes the velocities of bubbles of finite lengths in tubes, which are called slugs, provided their lengths exceed about one tube diameter.† Stewart (1965) examined all available experimental evidence on slug velocities and concluded that Dumitrescu's result was confirmed. He also performed a relaxation analysis to determine the shape of the slug at its nose assuming its velocity to be given by (2).

Another theoretical result for Dumitrescu's problem is due to Layzer (1955) who found

$$U_s = (gb/k_1)^{\frac{1}{2}} = 0{\cdot}511(gb)^{\frac{1}{2}}, \tag{5}$$

where $k_1 = 3{\cdot}8317\ldots$, which is the first positive zero of the Bessel function J_1. The three solutions for the slug limit given here, differ in the manner in which the condition of constant gas pressure is satisfied. Layzer's result was obtained in the spirit of what would now be called the first approximation so that his result would be expected as an asymptote at the slug limit in the present analysis. We shall, nevertheless, find it possible to use Dumitrescu's arguments to truncate the solution obtained here so that it agrees closely with his own result in (2), and thus agrees with the available experimental evidence at the slug limit.

2. Theory

In I, it was shown that the velocity of a large gas bubble in a liquid, is related to the radius of curvature *at* its front stagnation point, a, through the expression

$$U = (ga)^{\frac{1}{2}}/\{dh/d\theta\}_{\theta=0}, \tag{6}$$

† Davies & Taylor's experiments were performed by emptying long liquid-filled tubes by the sudden opening of their lower ends, while Dumitrescu injected very long slugs into his tubes.

where the angular co-ordinate θ is measured from an origin moving with the bubble and at the centre of curvature of the bubble at the front stagnation point, and where the irrotational velocity distribution on a model of the bubble surface is given in magnitude by

$$q = Uh(\theta). \tag{7}$$

It is convenient in the present axisymmetric problem to work with a system of cylindrical polar co-ordinates ϖ, x moving with the bubble and having their origin at the front stagnation point with x directed vertically upwards. In this co-ordinate system $(dh/d\theta)_{\theta=0}$ may be evaluated using the relation

$$\left\{\frac{dh}{d\theta}\right\}_{\theta=0} = \frac{1}{U}\left\{\frac{dq}{d\theta}\right\}_{\theta=0} = \frac{a}{U}\left\{\frac{dv}{d\varpi}\right\}_{\varpi=0}, \tag{8}$$

where v is the local velocity on the bubble surface in the direction of ϖ.

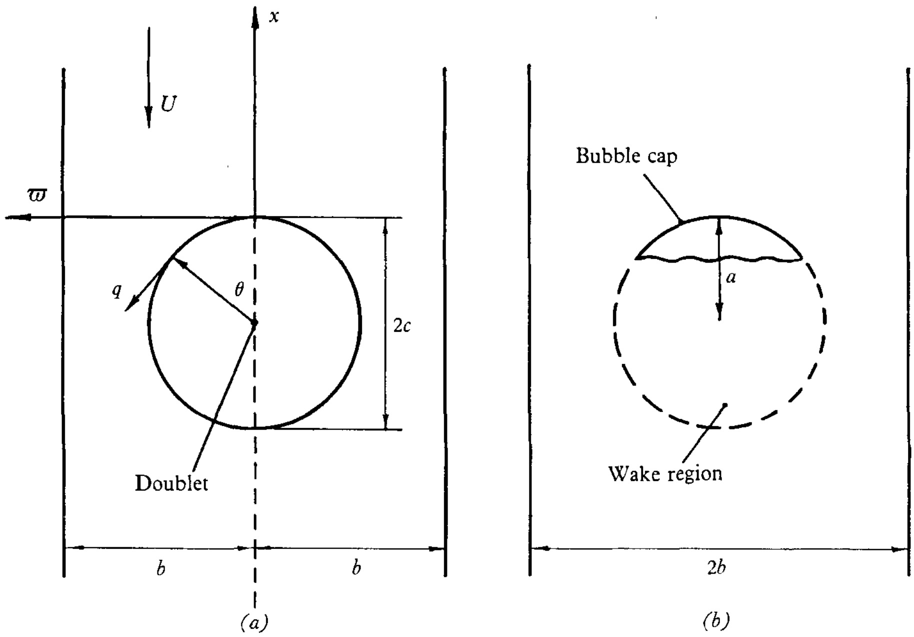

FIGURE 2. The flow system and its model.

In a previous paper (Collins 1965, II) it has been shown that a first approximation for the velocity of a large plane gas bubble moving along the axis of a vertical channel can be derived from the complex potential for a two-dimensional doublet within the channel. In this paper the corresponding axisymmetric irrotational flow will be used to construct the equivalent three-dimensional solution. The real bubble and its model are represented in figure 2. The radius of curvature of the bubble and its model at the forward stagnation point is denoted by a, the radius of the cylinder by b, and the length of the body forming the model and defined by the closed streamline is $2c$. This closed streamline is taken to represent the bubble shape in the region of the forward stagnation point and also models the wake behind the bubble to a certain extent.

An expression for Stokes's stream function for the flow of a uniform stream past a doublet on the axis of a cylindrical channel, which is valid for $x > -c$, may be deduced from a paper by Lamb (1926). If written in the modified form

$$\psi = U\left[\frac{\varpi^2}{2} - b\frac{\sum_n \{\varpi J_1(k_n\varpi/b)\exp[-k_n(x+c)/b]\}/J_0^2(k_n)}{\sum_n \{k_n \exp[-k_n c/b]\}/J_0^2(k_n)}\right], \tag{9}$$

then the doublet is situated at the point $x = -c$, and the doublet strength is adjusted by the function

$$\sum_n \{k_n \exp[-k_n c/b]\}/J_0^2(k_n)$$

so that the forward stagnation point is always at the origin of co-ordinates for all values of c/b. In equation (9), J_1 and J_0 are the familiar Bessel functions, and the summations are taken over all positive values of k_n for which $J_1(k_n) = 0$. The equation of the closed streamline $\psi = 0$ from (9) is

$$2b\sum_n \{J_1(k_n\varpi/b)\exp[-k_n(x+c)/b]\}/\varpi J_0^2(k_n) = \sum_n \{k_n \exp[-k_n c/b]\}/J_0^2(k_n). \tag{10}$$

In evaluating its radius of curvature at the origin it is convenient first to expand the function $\{J_1(k_n\varpi/b)\}/\varpi$ as a power series. On taking the second derivative, the radius of curvature is found as

$$\frac{a}{c} = 4\frac{b}{c}\frac{\sum_n \{k_n^2 \exp[-k_n c/b]\}/J_0^2(k_n)}{\sum_n \{k_n^3 \exp[-k_n c/b]\}/J_0^2(k_n)}. \tag{11}$$

It will prove to be convenient to have a shorthand notation for the type of function appearing on the right-hand side of (11). The role of these functions is to transfer control of velocity and shape from the parameter c to the cylinder dimension b. They are accordingly denoted by the functional symbol $T_{\alpha,\beta}$ defined so that

$$T_{\alpha,\beta}(c/b) = \frac{\sum_n \{k_n^\alpha \exp[-k_n c/b]\}/J_0^2(k_n)}{\sum_n \{k_n^\beta \exp[-k_n c/b]\}/J_0^2(k_n)}, \tag{12}$$

in which the summations are, as before, taken over the positive zeros of J_1. (11) thus appears as

$$a/c = 4(b/c)T_{2,3}(c/b). \tag{13}$$

One of the asymptotes of this equation is readily found without computation, for from (12), as $c/b \to \infty$, $T_{\alpha,\beta}(c/b) \to (k_1)^{\alpha-\beta}$, where $k_1 = 3{\cdot}8317\ldots$, is the first zero of J_1. From (13) therefore, $a/b \to 4/k_1 = 1{\cdot}044$ as $c/b \to \infty$, whereas in II the limiting value for the plane case was $a/b \to 3/\pi = 0{\cdot}955$.

To evaluate the other limit of equation (13) as $c/b \to 0$ one would strictly need a series expansion for $T_{\alpha,\beta}(c/b)$ valid for small c/b. A rigorous mathematical evaluation of this series, however, is unnecessary since for our purpose the asymptotic behaviour can be inferred using physical arguments. We expect $a/c \to 1$ as $c/b \to 0$ because the flow field is then infinite in extent and the closed streamline is then spherical and of radius a. This implies that $T_{2,3}(c/b) \sim c/4b$, which is confirmed by numerical evaluation performed on an electronic computer. Forty terms in the series were usually sufficient for this computation except for $c/b < 0{\cdot}2$ where convergence is very slow. The resulting form for a/c, drawn in figure 3, has the same character as in the plane case. The radius of curvature a, is sensibly constant and equal to c for values of $c/b < 0{\cdot}5$; when $c/b > 2{\cdot}5$, $a/b = 4/k_1 = 1{\cdot}044$. Some numerical values for a/c as a function of c/b are given in table 1.

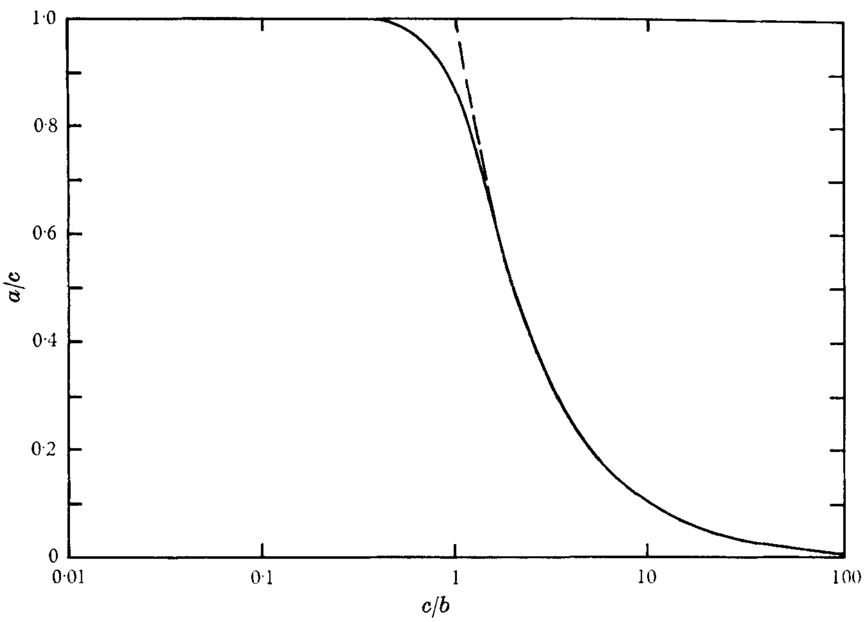

FIGURE 3. The radius of curvature of the model at its front stagnation point. ——, equation (11); – – –, $a/c = 1{\cdot}044b/c$.

c/b	0·1	0·2	0·3	0·45	0·6	0·8
a/c	1·000	1·000	0·999	0·993	0·977	0·931
c/b	1·0	1·5	2·0	4·0	6·0	10·0
a/c	0·863	0·668	0·518	0·261	0·174	0·104

For $c/b > 2{\cdot}5$, $a/c = 1{\cdot}044b/c$.

TABLE 1. Values of a/c as a function of c/b from equation (11)

By expanding the functions $\exp(-k_n x/b)$ and $J_1(k_n \varpi/b)$ in equation (9) as power series and differentiating to find the velocity on the ϖ-axis in the vicinity of the origin, it may be shown that

$$2(dh/d\theta)_{\theta=0} = (a/b)\,T_{2,1}(c/b), \tag{14}$$

from which it follows on using (6) that bubble velocity is

$$U = 2(ga)^{\frac{1}{2}}\,(b/a)\,T_{1,2}(c/b), \tag{15}$$

or alternatively

$$U = 2(gb)^{\frac{1}{2}}\,(b/a)^{\frac{1}{2}}\,T_{1,2}(c/b). \tag{16}$$

From (16) using (13) it is found that, as $c/b \to \infty$, $U \to U_s = (gb/k_1)^{\frac{1}{2}}$, agreeing with Layzer's result in equation (5). Again using physical arguments, since the closed streamline is spherical when $c/b \to 0$, then $a = \bar{a}$ and, as Davies & Taylor derived their result from consideration of this flow, one expects $U \to U_\infty = \frac{2}{3}(g\bar{a})^{\frac{1}{2}}$ as $c/b \to 0$. This implies $T_{1,2}(c/b) \sim c/3b$, which is again confirmed by numerical evaluation.† Equations (15) and (16) are shown as functions of a/b in figure 4 through the use of (13). The functions again have the same character as in the plane case. Comparison of figures 3 and 4 with their plane counterparts in II does show however, that the presence of the containing cylinder does not influence the velocity and shape of the bubble model as strongly as does the wall in the plane case. Table 2 gives a few values of U/U_∞ and $U/(gb)^{\frac{1}{2}}$, for various values of a/b.

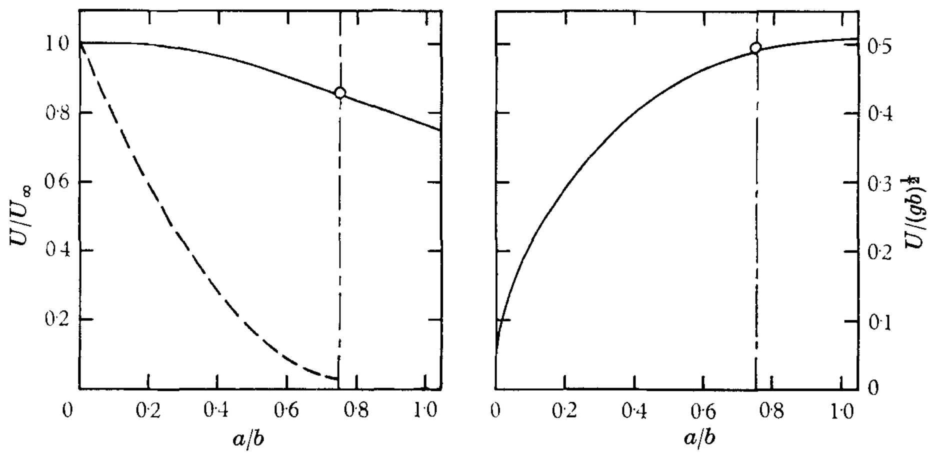

FIGURE 4. Variation of bubble velocity with a/b: ——, present theory; O, Dumitrescu's theory; ---, wall correction for solid spheres in Stokes flow.

It is interesting also to compare the boundary effect at the other extreme of Reynolds number. Very small bubbles in liquids are known to behave like solid bodies in the terminal velocities which they adopt, so that the cylindrical boundary affects their velocities in the same way as it would affect a solid sphere. Solutions for this problem have been fully discussed by Happel & Brenner (1965) and values for U/U_∞ taken from their tables are also plotted in figure 4; in this context, of course, U_∞ is the terminal velocity of the bubble in an infinite liquid, assuming Stokes flow. The behaviour of the two solutions is radically different. For example, at $a/b = 0{\cdot}1$, the effect of the boundary on a large gas bubble is negligible, while the very small bubble has its velocity reduced by more than 20 %.

The broken vertical lines in figure 4 are drawn at $a/b = 0{\cdot}75$. It will be shown in § 3.2 that there are plausible theoretical arguments, which were also employed

† Since $T_{1,2}(c/b)\,T_{2,3}(c/b) = T_{1,3}(c/b)$ it is concluded that $T_{1,3}(c/b) \sim c^2/12b^2$. A guess at the asymptotic behaviour in the general case on this rather meagre evidence is

$$T_{\alpha,\beta}(c/b) \sim \frac{(\alpha+1)!}{(\beta+1)!}\left(\frac{c}{b}\right)^{\beta-\alpha}.$$

by Dumitrescu, for truncating the present solution at that value. That is, one would not expect values of a/b exceeding 0·75 to be realised physically, so that the present solution above that value is discarded. Empirically an upper limit is found at $a/b = 0{\cdot}71$.

a/b	U/U_∞	$U/(gb)^{\frac{1}{2}}$	a/b	U/U_∞	$U/(gb)^{\frac{1}{2}}$
0·1	0·999	0·211	0·71	0·865	0·486
0·2	0·994	0·296	0·75	0·851	0·492
0·3	0·982	0·359	0·8	0·833	0·497
0·4	0·962	0·405	0·9	0·798	0·505
0·5	0·935	0·441	1·0	0·764	0·510
0·6	0·903	0·466	1·044	0·750	0·511
0·7	0·869	0·485			

TABLE 2. Values of U/U_∞ and $U/(gb)^{\frac{1}{2}}$ as functions of a/b from equations (15) and (16)

3. Modifications

(1) *Replacement of a by $\bar{a}$*

In deriving the first approximation for the infinite liquid limit, Davies & Taylor assumed that the flow over the bubble cap could be modelled by the irrotational flow over the forward part of a sphere having the same apparent curvature as the bubble. That is, they assumed that the radius of curvature of the bubble cap was constant so that $a = \bar{a}$. Because the present flow model reduces to theirs in the infinite fluid limit it was possible to make the same replacement at that limit in § 2. It is now necessary to justify the replacement of a by $\bar{a}$ over the whole range of a/b, for what is always measured in experiment is $\bar{a}$ and not a. In order to do this it must be shown that the differences between these quantities is small during transfer of control from c to b. This would be a difficult task in the axisymmetric case because the closed streamline given by equation (10) can be drawn only after lengthy iteration on to every point. Instead it will be demonstrated that $\bar{a}$ and a are very close in the plane case previously discussed in II, where the boundary is more readily drawn. This may be further regarded as belated justification for the same replacement implicit in II. At that time the second approximation, which points to the necessity for this justification, had not been obtained. Figure 5 compares the circle of radius equal to the radius of curvature at the front stagnation point, S, with the true theoretical boundary, for two values of c/b. Over a subtended angle of 75°, which was the angle matched by Davies & Taylor, there is no detectable difference for values of c/b up to 2·5. Transfer of control is complete at that value in the plane case because a/b has sensibly reached its limiting value of 0·955. The replacement of a by $\bar{a}$ over the whole range is thus justified because the influence of the wall in modifying the forward region of the bubble model, is delayed until after the limiting value of a/b has already been attained. There is no reason to expect any radically different behaviour in the axisymmetric case, indeed the fact that the effect of the cylinder is in general weaker than the effect of the wall may be used to argue that the delay in the axisymmetric case would be even longer.

(2) *Truncation of the solution*

We have seen that the analyses of Layzer and Dumitrescu differ slightly in their predictions of $U_s/(gb)^{\frac{1}{2}}$ for the slug. They differ also in another important respect, namely, in the value of a/b given for the slug. Layzer's model has $a/b = 1{\cdot}044$,

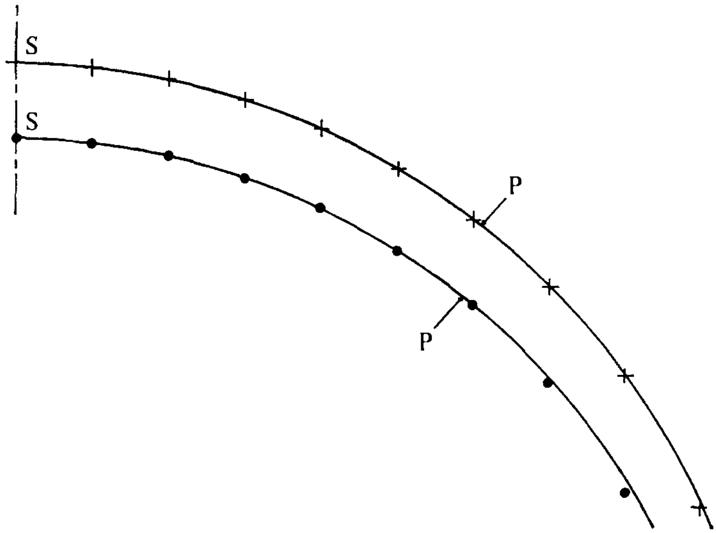

FIGURE 5. Comparison of plane bubble shapes with circles having their stagnation point curvature. $+$, $c/b = 1/\pi$, $a/c = 0{\cdot}994$, $a/b = 0{\cdot}316$; $\bullet$, $c/b = 2{\cdot}5$, $a/c = 0{\cdot}382$, $a/b = 0{\cdot}955$. The arc SP subtends an angle of $37{\cdot}5°$ at the apparent centre of curvature in each case.

which is the same as in the present analysis in the limit $c/b \to \infty$, while Dumitrescu's method gives $a/b = 0{\cdot}75$. Nothing can be done to correct Layzer's own value which is an inherent feature of the velocity potential which he assumed for the flow; a study of Dumitrescu's approach, however, indicates a plausible reason for truncating the present solution at $a/b = 0{\cdot}75$.

Having formulated the problem in a general manner, arguing on dimensional grounds that $U_s/(gb)^{\frac{1}{2}}$ should be a constant, λ, Dumitrescu constructed an approximate solution to his problem in the following way. He assumed the form of the bubble near the origin to be spherical† and of radius $\bar{a}$, which allowed him to derive a velocity potential for this assumed geometry, and hence to calculate λ as a function of $\bar{a}/b$. His method, however, allowed an arbitrary assignment of $\bar{a}/b$, so that he was then faced with the task of selecting the correct value. To do this he argued that the assumed boundary near the origin must be capable of being matched with the asymptotic form at large (negative) x. Since the liquid must be falling freely in that region, its mean velocity was readily determined in terms of its distance below the front stagnation point. A simple application of the equation of continuity gave the asymptotic form of the bubble as

$$\frac{x}{b} = \frac{-\lambda^2}{2\{1-(\varpi/b)^2\}^2}, \tag{17}$$

† In Dumitrescu's solution then, $a = \bar{a}$ exactly. The origin of the co-ordinates was the same.

which was found to merge imperceptibly with the spherical cap at $x = -0{\cdot}5b$, $\varpi = 0{\cdot}71b$ if $\bar{a}/b = 0{\cdot}75$, for which value $\lambda = 0{\cdot}496$. Now, if instead of regarding the stream function given by equation (9) to be a valid representation of the slug flow in the limit $c/b \to \infty$,† it is regarded as an approximation to the flow in the neighbourhood of the front stagnation point of a slug of undetermined a/b, then the limiting value of a/b may be determined using Dumitrescu's matching procedure. The only difference is that whereas Dumitrescu assumed the bubble form and then derived the flow, we have assumed the flow and calculated the form. In view of what has been said in the previous section about the replacement of a by $\bar{a}$, the results of these two approaches will be close, so that the present solution is also considered to be truncated at $\bar{a}/b = 0{\cdot}75$. At this value, $U_s/(gb)^{\frac{1}{2}} = 0{\cdot}492$, agreeing very well with equation (2). The truncation is indicated in figure 4.

In passing it can be mentioned that the same argument applied to the plane case discussed in II truncates that solution at $\bar{a}/b = 0{\cdot}72$ and adjusts the slug velocity to $U_s = 0{\cdot}321(gb)^{\frac{1}{2}}$. This is closer to Birkhoff & Carter's (1956) result than to Garabedian's (1957).

4. Experiments

Since Uno & Kintner's (1956) experimental paper on this topic is already widely referred to in the literature, it is necessary to explain why it is not possible to use their results to test the theory. First, although their paper has been frequently used to correct for the effect of the boundary on large gas bubbles, it must be used with caution for this purpose, since many of the bubbles used in their experiments do not fall into this class but belong to the class of ellipsoidal bubbles intermediate between spherical and spherical-cap bubbles. The authors correlated the ratio U/U^*_∞ with D_e/D, where U is the velocity of a bubble of volume V in a tube of diameter D, U^*_∞ is the velocity this *volume* of gas would assume in an infinite liquid, and D_e is the so called 'equivalent diameter', that is the diameter of a sphere having the volume V. As the theory cannot relate velocity with volume, it is not possible to use their results in this form for the purposes of comparison. Their paper does contain a figure purporting to correlate U/U^*_∞ with what is effectively $\bar{a}/b$. Apart from the fact that the ordinate of this figure still requires a knowledge of the relation between volume and velocity, it is suspect on other grounds. Unfortunately the authors did not once measure $\bar{a}$, but calculated values using their own measurements of volume and a 'compromise curve' relating $\bar{a}$ and equivalent radius, which was drawn to link up Rosenberg's data for spherical-cap bubbles with his data for ellipsoidal bubbles. As far as can be judged, this curve was employed for all the boundaries used in their experiments (which had diameters ranging from 2 to 15 cm) so that there was no allowance for the effect of the boundary on the shape adopted by a given volume of gas. There is the added complication that their figure apparently includes the ellipsoidal bubbles generated.

† This, of course, gives a stream function equivalent to Layzer's velocity potential.

A tank whose horizontal cross-section was a rectangle 16 in. × 13 in. and two of the vertical walls of which were of transparent acrylic sheet $\frac{1}{2}$ in. thick, was filled with water to a depth of 4 ft. In the base of the tank a circular recess 12 in. in diameter was turned so as to receive the insert shown in figure 6. Three circular grooves turned concentrically in the base board of this assembly served to locate

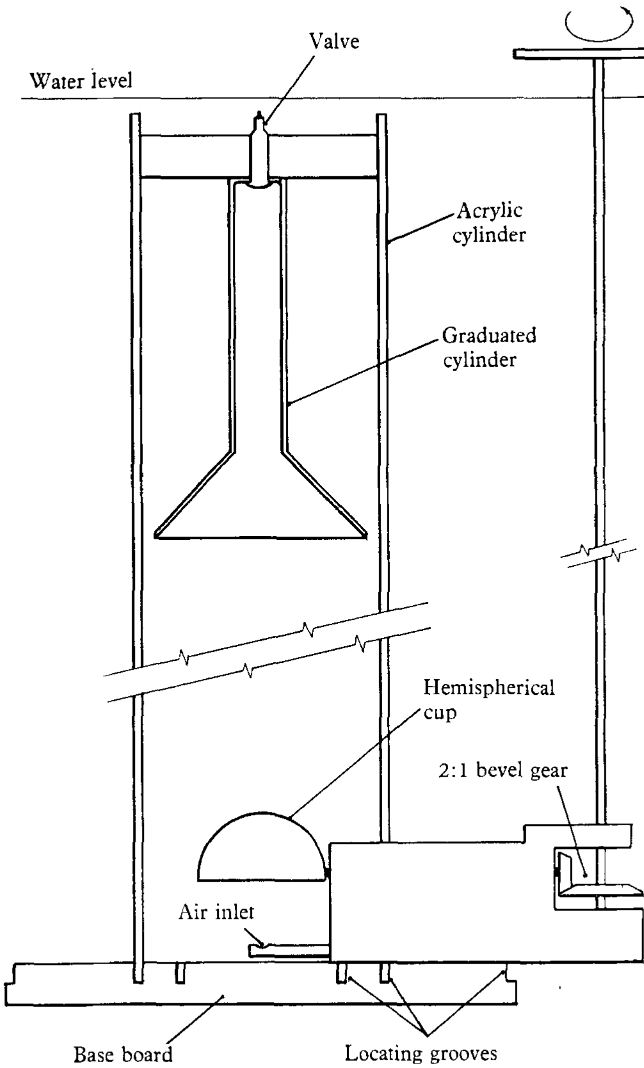

FIGURE 6. The tank insert.

the cylindrical acrylic inserts which formed the containing boundaries in the experiment. The bubbles were generated just above the centre of the board by the hemispherical cup technique employed by previous workers including Rosenberg and Uno & Kintner. A thin stainless steel hemispherical shell 3 in. in diameter was supported so that it could be rotated about a horizontal diameter in the plane of its rim. Air was introduced and trapped inside this inverted cup so that, when rotated, a spherical-cap bubble was produced near to the base of the tank and on the axis of the cylinders. By adjusting the rate of rotation it was possible to minimize the production of secondary bubbles but rarely possible to eliminate them completely. Each bubble was collected in a graduated cylinder at the top of the tank in order to determine its volume. The collection of the secondary bubbles at the same time constitutes an unavoidable but slight error in the measurement of volume.

Bubbles were photographed with a 35 mm 'Robot' camera which incorporated a motorized film transport mechanism. Two photographs of a bubble were taken separated in time by approximately $\frac{1}{4}$ sec, each photograph containing a scale and a clock so that bubble velocity could be determined. The initial photograph was taken when the bubble had reached the mid-point of its travel from generator to collector, a distance of approximately 3 ft. Bubble curvature was measured by projecting an enlarged image of the bubble on to a screen and comparing with a standard set of circular arcs which were matched to the bubble over an included angle of 75°.

Three acrylic cylinders having internal diameters 3·6 in., 7·6 in. and 11·6 in., formed the cylindrical boundaries and a range of bubbles was blown in each. Measurements were taken from sixty-one bubbles in all. Detailed experimental results from the present and from previous investigations have been recorded separately (Collins 1967).

5. Discussion

The experimental results are compared with the theory in figure 7, which also includes the results of Dumitrescu, Rosenberg and those of Davies & Taylor relating to bubbles in nitrobenzene. Davies & Taylor's experiments were performed in a tank whose horizontal cross section was a square of side 2 ft. In order to plot the results, an equivalent dimension b for this system was taken as the radius of a circle having an area 4 ft^2, that is $b = 1{\cdot}13$ ft. Rosenberg's tank was large, but of unusual cross-section 4·5 ft. × 25 ft., the experiments being performed at one end. An equivalent $b = 2{\cdot}54$ ft. for his experiments was calculated in a similar manner assuming the tank to be effectively square of side 4·5 ft. Although these equivalent dimensions appear crude, in neither case are they critical, since the tanks were sufficiently large to preclude wall effect anyway. Since no tabulated values were given in Rosenberg's paper, values of U and $\bar{a}$ for his experiments were taken from an enlarged image of one of his figures projected on to a suitably scaled grid. There is the possibility of a slight error in replotting by this method.

The agreement of the experiments with the theory is good. If anything, the experiments fall slightly below the theoretical line but this is in agreement with our knowledge that, at the infinite liquid limit, the first approximation is approximately 2 % above a second, which is also shown on the figure. The upper limit of $\bar{a}/b$ appears to be $\bar{a}/b = 0{\cdot}71$, slightly lower than the theoretical limit given by Dumitrescu's matching procedure. It is interesting to note that Dumitrescu's experimental shape also gives $\bar{a}/b = 0{\cdot}71$, although this was not quoted as one of his main results. At this value Dumitrescu's theoretical velocity would be $U_s = 0{\cdot}49(gb)^{\frac{1}{2}}$ while the present theory gives $U_s = 0{\cdot}486(gb)^{\frac{1}{2}}$. Stewart's relaxation computations gave a limiting value of $\bar{a}/b = 0{\cdot}71$ at a velocity $U_s = 0{\cdot}496(gb)^{\frac{1}{2}}$; they also showed that $a = \bar{a}$ over an included angle of 75°.

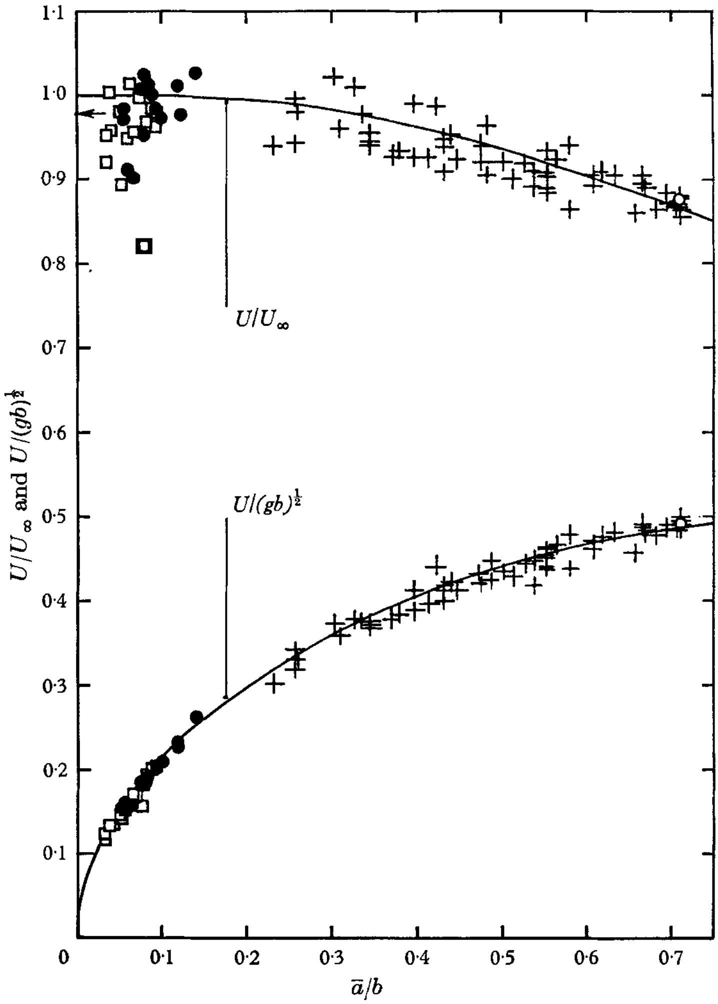

FIGURE 7. Comparison of theory with experiment: ←, asymptote for second approximation; ——, present theory; +, present experiments; □, Rosenberg's experiments; ○, mean of Dumitrescu's experiments; ●, Davies & Taylor's experiments in nitrobenzene.

Figure 8 correlates $\bar{a}/b$ with $V^{\frac{1}{3}}/b$, the scatter being indicative of a lack of geometrical similarity of bubbles of given $V^{\frac{1}{3}}/b$. Davies & Taylor's nitrobenzene results are again included. Drawn through the points and apparently correlating them well, is a line having the equation

$$\bar{a}/b = 0{\cdot}71 \tanh^{\frac{1}{2}}\{4{\cdot}25(V^{\frac{1}{3}}/b)^2\}. \tag{18}$$

There is no theoretical basis for this equation, but it does appear to possess the appropriate shape together with appropriate asymptotes. At one extreme, as

$V^{\frac{1}{3}}/b \to \infty, \bar{a}/b \to 0{\cdot}71$, but the asymptote is approached so rapidly that $\bar{a}/b = 0{\cdot}706$ when $V^{\frac{1}{3}}/b = 0{\cdot}8$. The asymptote of (18) as $b \to \infty$ is

$$\bar{a} = 1{\cdot}465 V^{\frac{1}{3}}. \tag{19}$$

To examine the suitability of this result we may employ the empirical relation derived by Davies & Taylor relating bubble velocity with volume. Augmenting their nitrobenzene results with results from a large number of air bubbles in water (for which $\bar{a}$ was apparently not measured) they deduced a result which may be written as

$$U^*_\infty = 0{\cdot}792(gV^{\frac{1}{3}})^{\frac{1}{2}}. \tag{20}$$

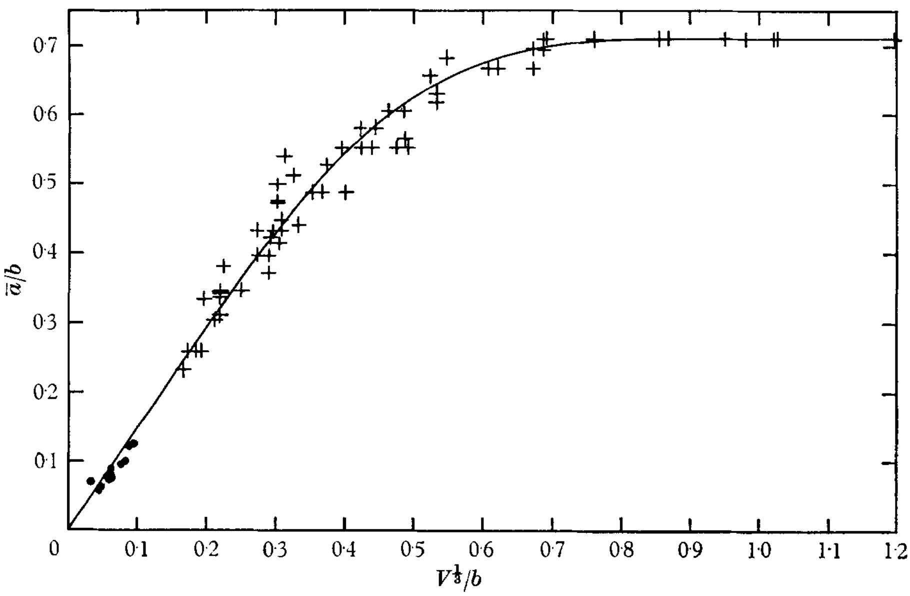

FIGURE 8. Variation of apparent curvature with bubble volume: +, present experiments; ●, Davies & Taylor's experiments in nitrobenzene; ——, equation (18).

On using (19), this result may be recast in terms of $\bar{a}$ as

$$U_\infty = 0{\cdot}654(g\bar{a})^{\frac{1}{2}}. \tag{21}$$

This result, rather circuitously obtained, lies between the first and second approximations at the infinite liquid limit and is somewhat closer to the second than the first. It is also very close to the mean of the combined results of Rosenberg and Davies & Taylor's nitrobenzene experiments, which is $U_\infty = 0{\cdot}65(g\bar{a})^{\frac{1}{2}}$. The asymptotic behaviour of (18) is thus consistent with previous work.

Having established the theory as a good first approximation, the experimental results may be replotted in a form comparable with Uno & Kintner's. Figure 9 contains the available experimental evidence and shows U/U^*_∞ as a function of $V^{\frac{1}{3}}/b$.† In addition to the present results, this figure includes results from several other investigations. Those for bubbles in water used by Davies & Taylor in establishing equation (20) are shown, values for these points being taken from

† For those more accustomed to the use of the equivalent diameter D_e as a measure of bubble size, a subsidiary axis indicates the range of variation of $D_e/2b$.

an enlarged figure in the original report from which their paper was derived. Unlike the bubbles in nitrobenzene, these bubbles were blown in a cylindrical tank 2 ft. 6 in. in diameter. Results from Uno & Kintner's experiments plotted in figure 9 relate only to those bubbles which can be considered of spherical-cap form, that is having volumes in excess of about 3·5 cm^3. These bubbles were formed in water, diethylene–glycol and a glycerine–water mixture in tubes whose diameters ranged from 3·64 to 15·25 cm. The values came from their tables

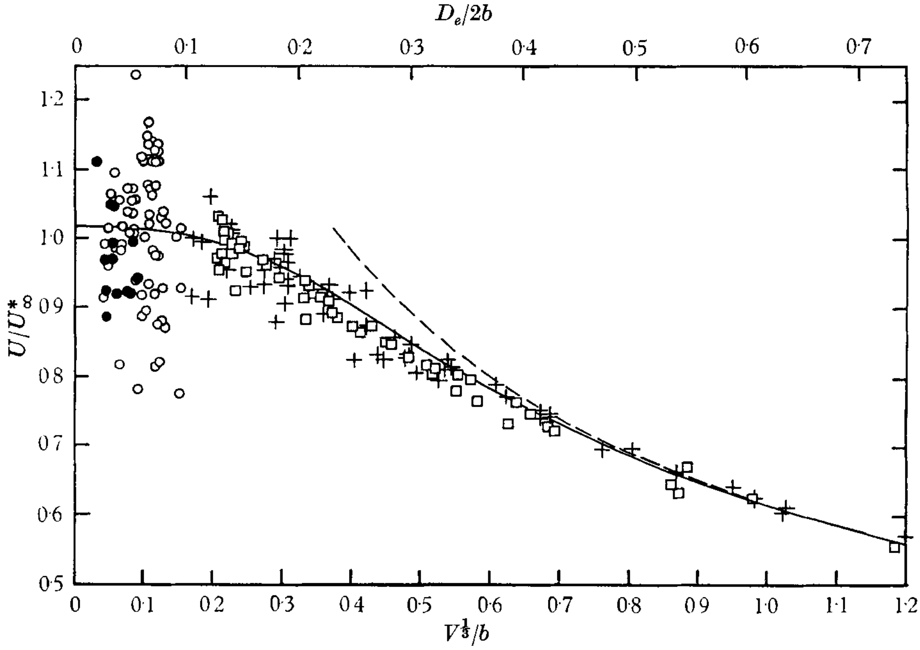

FIGURE 9. Variation of U/U^*_∞ with $V^{\frac{1}{3}}/b$: ——, semi-empirical line; – – –, equation (23); +, present experiments; ●, Davies & Taylor's experiments in nitrobenzene; ○, Davies & Taylor's experiments in water; □, Uno & Kintner's experiments.

published separately, and not from the smoothed curves presented in their paper. The values of U^*_∞ used in plotting all results in this figure were those given by (20). In some cases this does not agree exactly with Uno & Kintner's estimate, but the differences are slight.

The continuous line labelled 'semi-empirical' on this figure results from combining the theoretical velocity given by equations (15) and (16) with the empirical results contained in equations (18) and (20). The scatter of the points is high, particularly at low values of $V^{\frac{1}{3}}/b$, but this line, none the less, correlates the results quite well. It is a little curious at first sight that it should give $U/U^*_\infty = 1{\cdot}018$ at $V^{\frac{1}{3}}/b = 0$, but this is merely a reflection of the fact that U is a theoretical first approximation and is therefore slightly high at that asymptote, while U^*_∞ is a purely empirical quantity.

Also shown in figure 9 is the asymptote at the slug limit which comes from combining the present theoretical slug velocity

$$U_s = 0{\cdot}492(gb)^{\frac{1}{2}} \tag{22}$$

with the empirical equation (20) to give

$$U_s/U_\infty^* = 0{\cdot}62(b/V^{\frac{1}{3}})^{\frac{1}{2}}. \qquad (23)$$

Three régimes of flow may thus be discerned in this figure. When $V^{\frac{1}{3}}/b < 0{\cdot}15$ the effect of the containing cylindrical boundary is negligible; there is a transition region in which $0{\cdot}15 < V^{\frac{1}{3}}/b < 0{\cdot}8$; finally, when $V^{\frac{1}{3}}/b > 0{\cdot}8$, the bubble behaves effectively as if it were a slug. From figure 8, the limiting value of a/b has then been sensibly reached.

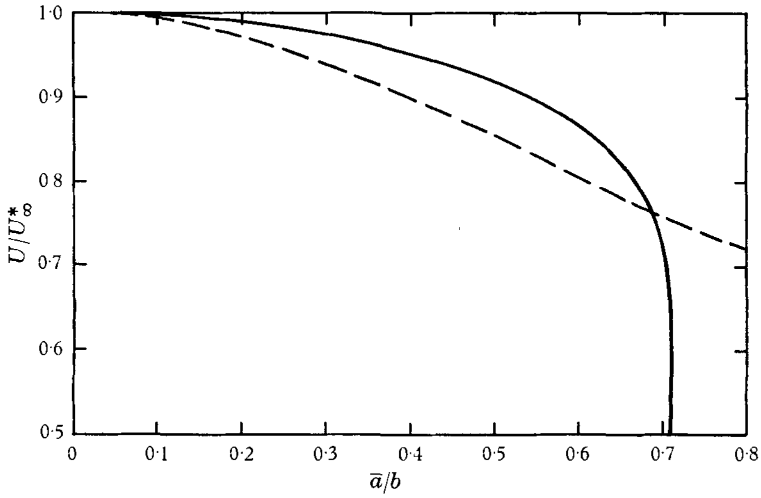

FIGURE 10. Variation of U/U_∞^* with $\bar{a}/b$: ——, semi-empirical line; – – –, mean line through Uno & Kintner's data in their figure 10.

There is no agreement on comparing the present results with Uno & Kintner's figure 10 purporting to show U/U_∞^* as a function of $\bar{a}/b$. The comparison is effected in figure 10 here. Undoubtedly the difference in form is due to their use of a 'compromise curve' to obtain $\bar{a}/b$ from their measured volumes. We note, for example, the prediction of $\bar{a}/b$ as high as 0·8, whereas the maximum value recorded here and by other writers is 0·71. It is concluded that figure 10 of Uno & Kintner's paper is invalid.

Further discussion of the application of the present results to gas bubbles in fluidized beds, together with some suggested empirical relations, may be found in the report previously cited (Collins 1967).

To summarize, a theoretical first approximation for the velocity of a large gas bubble rising along the axis of a liquid-filled circular cylinder provides an adequate description of the behaviour of a real bubble. This theory links the previously known asymptotes at the infinite liquid and slug limits. It may be recast as a semi-empirical theory relating velocity with bubble volume, which agrees well both with the present experiments and with those of previous writers.

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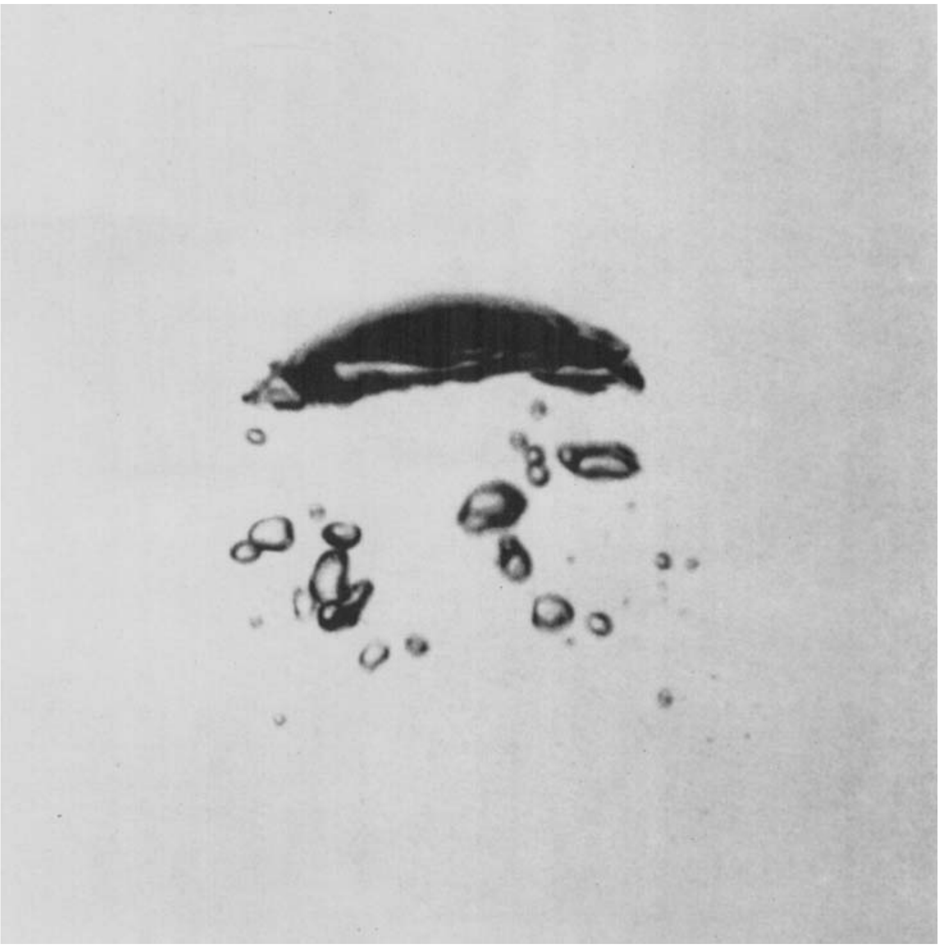

FIGURE 1. The form of a large gas bubble in a liquid; $\bar{a} \simeq 1{\cdot}9$ in.